How to Earn Money in Live Streaming Platforms? — A Study of Donation-Based Markets

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Abstract*—Donation-based markets are becoming increasingly popular in our daily life. One example is the online streaming platform Twitch, which attracts millions of users on a daily basis. On such platforms, firms provide services to customers without mandatory charge, and customers voluntarily donate money to the firms. The donations are split between the firms and the platform with a fixed pre-agreed fraction. To gain insights into the operation and optimization of such platforms, we formulate a two-stage game to study the platform's and firms' behaviors. In Stage I, the platform decides a donation-split-fraction (DSF), which corresponds to the fraction of donations kept by the firms. In Stage II, firms decide whether to participate in the platform and how to choose their service attributes considering the DSF as well as the preferences of firms and customers. Analyzing such a two-stage game directly is challenging, as the Stage II problem corresponds to the multi-firm extension of the Hotelling model and is still an open problem. To resolve this issue, we approximate the large number of firms as non-atomic decision makers, where a single firm's strategy choice does not affect the payoffs of the firm population. Under such an approximation, we prove that the Stage II problem is a potential game. We further show that at the equilibrium, a larger DSF leads to more firm participations and a better match to the customers' preferences. The stage I problem, nevertheless, is a non-convex optimization problem that does not render a closed-form solution. To gain insights, we derive the upper-bound and lower-bound of the optimal DSF solution. The bounds suggest that the platform should increase its DSF if the customers' donation sensitivity to the number of firms increases or if the firms' opportunity cost for participation increases. Finally, we collect data from Twitch and demonstrate the results of the two-stage model with a case study. Our simulation results suggest that under our data and model settings, there exists a significant potential for Twitch to improve its payoff, by setting the DSF to 0.38, instead of 0.71 as in Twitch's current practice.

I. Introduction

A. Background and Motivation

Recently, many online platforms have chosen to implement a donation-based market between two groups of users: firms who provide services without mandatory charges, and customers who enjoy the services and voluntarily donate to the firms. The customers donate mainly due to their desires of being acknowledged on the platforms (to gain community presents) and supporting the firms for future service provisions

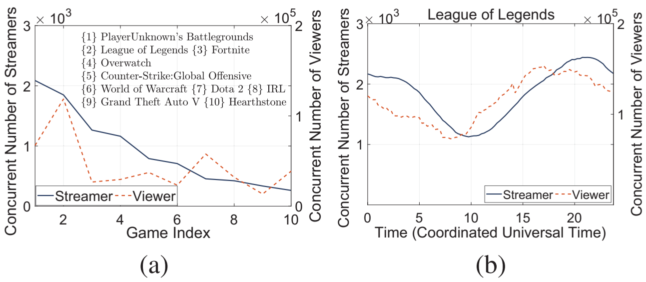

Fig. 1. Mismatch of the concurrent numbers of streamers and viewers in Twitch: (a) game attribute; (b) time attribute.

[1]. These donations are split between the firms and the platform with a fixed *donation-split-fraction* (DSF), which corresponds to the fraction of donations kept by the firms.

There are many examples of donation-based markets, such as several live streaming platforms, including Twitch[1]. In live streaming platforms, streamers (corresponding to firms) live stream their videos of game play, while viewers (corresponding to customers) watch these live streaming videos for free. The viewers can donate to the streamers, and a fixed fraction of the donations will be kept by the streamers (e.g., $1/1.4 \approx 0.71$ on Twitch). The total donation volume on similar types of platforms is huge. In 2017, a total of $\$101$ million dollars of donations were received by top live streaming platforms including Twitch, YouTube Live, Mixer, Facebook Live, and Periscope [2]. Other donation-based market examples are blogging platforms (e.g., WeChat Subscription) and online music platforms (e.g., Songtradr).[2]

The donation-based feature of these markets brings two unique questions as follows:

First, from the firms' point of view, *how should they decide their service attributes (e.g., in a live streaming platform, what game to broadcast at what time) given a fixed DSF?* The firms and customers may have different preferences over the service attributes, and firms' choices (which can be different from

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[1]Twitch (https://www.twitch.tv/) is the largest live streaming platform in the US [2]. It has more than 15 million unique daily visitors and 2 million unique monthly streamers, according to its annual report of 2017 [3].

[2]WeChat Subscription (https://mp.weixin.qq.com/?lang=en_US) is a blogging platform for individual article publishing. Songtradr (https://www.songtradr.com/) is a music platform for independent musicians.

their own preferences) will affect the competition levels among firms and the satisfactions of the customers.

As an example, Figure 1 illustrates the mismatch of the numbers of streamers and viewers in Twitch.[3] The subfigures (a) and (b) correspond to game and time attributes, respectively. Specifically, Figure 1(a) shows the average concurrent numbers of streamers and viewers (over a two-week data collection period) of different games. Some games with a small number of viewers have a large number of streamers (e.g., $\{3\}$ Fortnite), which implies that these streamers may improve their payoffs by switching to stream other games with less competitors. Figure 1(b) shows the corresponding average numbers (over the same two-week period) of the game *League of Legends* at different times. Similarly, some streamers may increase their payoffs by changing their streaming time (e.g., from 3am to 11am in Coordinated Universal Time)

Second, from the platform's point of view, *how should it set the DSF to maximize its payoff?* A higher DSF leads to a smaller per-donation revenue to the platform. On the other hand, it can increase the incentive for the firms to participate in the platform and better match the customers' preferences to induce more donations.

Despite the fact that the donation-based market has been embraced by top companies (e.g., YouTube and Facebook) and attracts millions of firms and customers, there does not exist a good understanding regarding the answers of the above two key questions. This motivates the research in this paper.

B. Solution Approach and Contribution

For the sake of concreteness, we focus on the example of the live streaming platform in this paper.[4] The modeling approach and analysis techniques are applicable to other donation-based markets as well. The platform first announces the DSF, then each firm decides whether to participate and what service attribute to choose (for example, at what time to stream). We model such a sequential decision process by using a two-stage game. This game is challenging to solve due to several reasons.

First, consider the Stage II problem where firms make their participation and service attribute selection decisions. This is an extended version of the Hotelling model [4] with many firms, which is still an open problem [5]. To resolve this issue, we consider a large population approximation where each firm is non-atomic, i.e., a single firm's strategy choice does not affect the entire market. This approximation is reasonable given the large number of firms (and customers) on these platforms in practice. The remaining difficulty is to compute the asymmetric equilibrium, where firms of the same preference may choose different strategies at the equilibrium. This is significant more difficult than focusing on the symmetric equilibrium only as in many previous works [6]. Despite these difficulties, we are able to prove that the Stage II game is a potential game [7], based on which we derive the game equilibria and corresponding equilibrium features.

Next, we consider the Stage I problem where the platform optimizes the value of DSF. The problem is non-convex and hence is challenging to solve. By exploiting the structure of the problem, we derive the upper-bound and lower-bound of the optimal solution and obtain practical insights.

Our key contributions are listed as follows:

- *Donation-Based Market Formulation:* To the best of our knowledge, this is the first paper that presents a two-stage model of a donation-based market. We characterize how the platform optimizes its DSF, and how the firms decide whether to participate and choose their service attributes.
- *Stage II Equilibrium of Firm Behavior:* For the Stage II problem, we consider a large population approximation with non-atomic firms. We prove that the Stage II problem is a potential game and derive the asymmetric equilibria. We show that a larger DSF leads to more firm participations and a better match to the customers' preferences at the equilibrium.
- *Stage I Equilibrium of Platform Behavior:* For the Stage I non-convex optimization problem, we derive the upper- and lower-bound of the optimal solution, which reflects how the optimal DSF changes with system parameters. As the bounds suggest, the platform should increase its DSF if the customers' donation sensitivity to the number of firms increases or if the firms' opportunity cost increases.
- *A Case Study based on Empirical Twitch Data:* We collect two weeks' data about streamers' and viewers' behaviors from the Twitch platform. Based on the data, we demonstrate how to compute the platform's optimal DSF without knowing the preferences of the firms and customers. The study suggests that under our data and model settings, Twitch should set the DSF to be 0.38, rather than the 0.71 in reality, to maximize its revenue.

The rest of this paper is organized as follows. We review the existing works in Section II. We propose the system model in Section III. In Sections IV and V, we analyze the equilibria of Stages II and I, respectively. We perform the case study with Twitch data in Section VI, and conclude in Section VII.

II. Literature Review

A. Donation-Based Market

Most of the prior works on donation-based markets studied the customers' donation behaviors. Hu *et al.* [8] conducted an online survey to study why customers visit live streaming platforms, where some of the reasons (such as cognitive communion and resonant contagion) also explain their donation behaviors. Scheibe *et al.* [1] conducted surveys to study why customers donate, and the major reasons include the customers' desires of being acknowledged on the platforms and supporting the firms for future service provisions. Zhu *et al.* [9] analyzed the data from Douyu (a live streaming platform in China) to investigate the customers' donation behaviors. Tang *et al.* [10] used an all-paid auction framework to understand the customers' donation behaviors.

[3] Figure 1 is based on the stream data that we collected from Twitch. The data is collected every 15 minutes from Nov. 05 to Nov. 20, 2017.

[4] We will use the pair of words "streamer" and "firm" and the pair of words "viewer" and "customer" interchangeably in this paper.

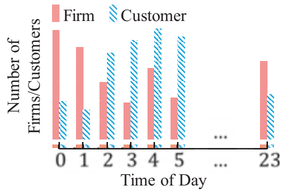


Fig. 2. System model: an example with time attribute.

Through data analysis, some papers identified the importance of motivating firm service selection behaviors. For example, Jia *et al.* [11] discussed the firms' and customers' different preferences on live streams, and mentioned that the platform has to motivate firms to participate and match the customers' preferences to increase the platform's revenue.

However, as far as we know, there is no paper analytically characterizing how the platform should motivate the firms' service selections. As a first step, this work studies the platform's optimal DSF decision and analyzes the firms' service selections in these donation-based markets.

B. Hotelling Model

The Stage II of the two-stage game can be regarded as an extended version of the Hotelling model [4], [12]. In the classical Hotelling model, customers are distributed along an interval (representing their preferences over an service attribute), and two firms decide their locations over the interval to maximize their own payoffs, respectively. The survey [5] presented recent extensions of the Hotelling model, such as different service attribute (e.g., over a line or a circle) and different firm decision process (e.g., simultaneous or sequential).

The Stage II model in our paper is related to a Hotelling model with a larger number of firms. This still remains as an open problem in literature [5]. Economides [13] studied a multi-firm model on the interval without discussing the firm equilibria. Brenner [14] theoretically studied a three-firm case, and empirically studied four- to nine-firm cases. Behringer *et al.* [15] theoretically analyzed a four-firm case. However, the analysis in [14] and [15] cannot be easily generalized to the case of an arbitrary number of firms.

We circumvent the difficulty by approximating the problem with a large number of non-atomic firms, where a single firm's strategy choice does not affect the market. Schmeidler [16] first analyzed a game with non-atomic players, and proved the existence of Nash equilibrium (without deriving the equilibrium). However, we are not aware of papers that explicitly characterizing the equilibrium of a general Hotelling model with non-atomic firms. Based on the reformulated model, we derive the asymmetric equilibria with non-atomic players, which is a challenging problem according to [6].

III. System Model

In this section, we first introduce the system setting, and then define the two-stage game.

A. System Setting

We first introduce the platform model and the service attribute. Then, we introduce the firm and customer models.

1) Platform: We consider a platform with a large number non-atomic firms and non-atomic customers, where the firm and customer sets are continuum. The large population setting is reasonable in practice, e.g., Twitch often has thousands of streamers and millions of viewers on average (see Figure 1).

The firms provide services without mandatory charge, and the customers enjoy the services and voluntarily donate to the firms. The donation will be shared between the firms and the platform with a fixed fraction. As the firms and customers are non-atomic, we will consider the aggregate donations from the customers and the average donations earned by each firm, with details discussed in Section III-A4.

2) Service Attribute: For simplicity, we only consider one service attribute in this paper. For example, on a live game streaming platform, the attribute can be the streaming time or the type of game to be streamed.

Similar as in the classical Hotelling model [4], we represent the attribute using a unit line segment of $[0,1]$. We label the possible values of the attribute, named as locations, by the set of $\mathcal{L}=\{0,1,2,...,L\}$, where the s^{th} location is located at $l_s \in [0,1]$ along the interval. Figure 2 shows an example, where we have a location set $\mathcal{L}=\{0,1,...,23\}$, each representing an hour of the day. Firms and customers are distributed along the interval based on their preferences, and the firms can decide their stream times over the set $\mathcal{L}$.

3) Non-Atomic Firms: A firm has a preferred location (i.e., value of the attribute). Let N^s denote the number of the firms preferring the location $s \in \mathcal{L}$. A firm can choose an attribute that is different from his preference, so as to avoid competitions with other firms or encounter more customers.

4) Non-Atomic Customers: A customer has a preferred location. Let M^s denote the number of customers preferring the location $s \in \mathcal{L}$. In this paper we focus on studying the decision of the firms. For simplicity, we assume that a customer always choose his preferred attribute.

A customer will donate to the firms whose selected attributes are the same as the customer's own preference. Instead of characterizing the donation behavior of each customer, we consider an aggregate donation function $D(M,N)$, depending on the number of firms N (according to the firms' choices) and the number of customers M (according to the customers' preferences) at one particular location. We assume that the donation function $D(M,N)$ satisfies the following assumption.

Assumption 1 (Donation Function). *Function $D(M,N)$ (i) is strictly increasing in M, (ii) is strictly increasing and concave in N, and (iii) has an elasticity that is smaller than one, i.e.,*

$$\eta_M(N)=\frac{[D(M,N)]_N\cdot N}{D(M,N)}\le 1,\quad \forall M,N\in\mathbb{R}_+, \tag{1}$$

where $[D(M,N)]_N$ denotes the partial derivative of $D(M,N)$ with respect to N.

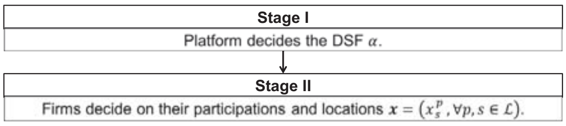


Fig. 3. Two-stage game.

Note that we do not characterize the customer donation behaviors directly, while we use a donation function $D(M,N)$ to represent the resulting donations. As a result, the analyses in this paper apply for any customer donation behavior as long as it induces a donation function in the form of $D(M,N)$ that satisfies Assumption 1. Specifically, in Assumption 1, point (i) implies that as the number of customers increases, the total donation strictly increases. Point (ii) implies that as the number of firms increases, the total donation strictly increases but the marginal change decreases. In live streaming platforms, for example, more streamers implies a higher probability that a viewer can find his satisfactory streams so that he will donate more, while the probability of finding a satisfactory stream is concave in the number of streamers. Point (iii) on elasticity can be written as follows:

$$1 \geq \frac{[D(M,N)]_N \cdot N}{D(M,N)} \approx \frac{\%\Delta D(M,N)}{\%\Delta N}, \ \forall M, N \in \mathbb{R}_+. \quad (2)$$

This implies that a unit percentage increase in the number of firms leads to a percentage donation increase less than one. Because of this, the firms tend to avoid competition (e.g., a streamer would prefer to stream at a time when there are more viewers and less streamers).

B. Two-Stage Game

1) Two-Stage Game: Let us take the live streaming platform as an example. The platform first announces the DSF, then each streamer decides whether to participate and what attribute to choose. Inspired by such a sequential decision process, we model the donation-based market using a two-stage game, as shown in Figure 3. Next, we explain the two stages in details.

In Stage I, the platform decides the DSF $\alpha \in [0,1]$, i.e., the fraction of donations kept by firms.

In Stage II, firms decide whether to participate and what will be their location choices (if they choose to participate). In this paper, we use superscripts to denote preferences and subscripts to denote decisions. Let x_s^p denote the number of firms preferring location $p \in \mathcal{L}$ and choosing location $s \in \mathcal{L}$. Note that we allow asymmetric equilibrium, hence x_s^p maybe positive for multiple values of s. Due to the assumption of non-atomic firms, the x_s^p can take a non-integer value. The strategies of the firms preferring location $p \in \mathcal{L}$ is characterized by $\boldsymbol{x}^p = (x_s^p, \forall s \in \mathcal{L})$. Let $\boldsymbol{x} = (x_s^p, \forall p, s \in \mathcal{L})$ denote the strategies of all the firms.

In addition, any strategy $\boldsymbol{x}$ should satisfy the constraint that for the firms preferring any location $p \in \mathcal{L}$, the total number of participating firms should be no larger than the total number of firms, i.e., $\sum_{s\in\mathcal{L}} x_s^p \leq N^p, \forall p \in \mathcal{L}$. For presentation simplicity, let $\widehat{N}_s(\boldsymbol{x}) \triangleq \sum_{p\in\mathcal{L}} x_s^p$ be the aggregate number of firms choosing location s under a strategy $\boldsymbol{x}$.

TABLE I
KEY NOTATIONS.

Parameters	
N^s	The number of firms preferring location $s \in \mathcal{L}$
M^s	The number of customers preferring location $s \in \mathcal{L}$
V	A firm's opportunity cost for participation
W	A firm's deviation cost per unit distance
Decisions & Decision-Related Notations	
α	Platform's DSF decision
x_s^p	The number of firms preferring $p \in \mathcal{L}$ and choosing $s \in \mathcal{L}$
$\boldsymbol{x}^p$	$\boldsymbol{x}^p = (x_s^p, \forall s \in \mathcal{L})$, the strategies of firms preferring $p \in \mathcal{L}$
$\boldsymbol{x}$	$\boldsymbol{x} = (x_s^p, \forall p, s \in \mathcal{L})$, the strategies of all firms
$\widehat{N}_s(\boldsymbol{x})$	$\widehat{N}_s(\boldsymbol{x}) = \sum_{p\in\mathcal{L}} x_s^p$, the total number of firms choosing location $s \in \mathcal{L}$ under strategy $\boldsymbol{x}$

2) Payoff Functions: Given the platform strategy α and the firm strategy $\boldsymbol{x}$, we define their payoffs as follows.

Platform's Payoff: The platform's payoff equals $1-\alpha$ fraction of the total donations from customers at all locations:

$$G(\alpha, \boldsymbol{x}) = (1-\alpha) \sum_{s\in\mathcal{L}} D(M^s, \widehat{N}_s(\boldsymbol{x})). \quad (3)$$

A Firm's Payoff: If a firm does not participate in the platform, it gains a zero payoff.[5]

If a firm preferring location $p \in \mathcal{L}$ participates and chooses a location $s \in \mathcal{L}$, its payoff equals the difference between its donation gain and its cost, i.e.,

$$F_s^p(\alpha, \boldsymbol{x}) = \alpha \times U(M^s, \widehat{N}_s(\boldsymbol{x})) - C_s^p(V, W), \ \forall p, s \in \mathcal{L}. \quad (4)$$

Specifically, the donation gain is the DSF α multiplied by the average donation that a firm can gain at the location s, where the average donation is defined as

$$U(M^s, \widehat{N}_s(\boldsymbol{x})) = \frac{D(M^s, \widehat{N}_s(\boldsymbol{x}))}{\widehat{N}_s(\boldsymbol{x})}, \quad (5)$$

where $U(M^s, \widehat{N}_s(\boldsymbol{x}))$ is increasing in M^s and decreasing in $\widehat{N}_s(\boldsymbol{x})$ according to Assumption 1. This shows that a firm gains a higher donation gain at a location with more customers or less firms. The cost contains a fixed opportunity cost V and a distance-associated deviation cost $W \times (l_p - l_s)^2$:

$$C_s^p(V, W) = V + W \times (l_p - l_s)^2. \quad (6)$$

The quadratic form of the deviation cost is used to characterize the firms' increasing marginal costs on the deviation, similar as in the original Hotelling model [4]. This shows that a firm consumes a higher cost if it has a higher opportunity cost or it chooses a longer deviation.

Table I summarizes the key notations of this paper. We solve the two-stage game using backward induction. Next, we analyze the Stage II equilibrium in Section IV and the Stage I equilibrium in Section V.

[5]If the non-participation induces a positive payoff, we can normalize it to zero by adjusting the value of the opportunity cost V defined in (6).

IV. STAGE II: FIRM LOCATION EQUILIBRIUM

In Stage II, given any DSF α, we analyze the firm location game as follows.

Definition 1 (Stage II Firm Location Game)**.**

- *Players: all the firms;*
- *Strategies: each firm preferring a location $p \in \mathcal{L}$ selects a location $s \in \mathcal{L}$, and the aggregate strategy is represented by $\boldsymbol{x} = (x_s^p, \forall p, s \in \mathcal{L})$;*
- *Payoffs: $F_s^p(\alpha, \boldsymbol{x})$ for each firm preferring a location $p \in \mathcal{L}$ and choosing a location $s \in \mathcal{L}$.*

The key objective in this section is to derive the firm location equilibrium, under which firms have no incentive to change their location choices. We first define the equilibrium, and then derive the equilibrium and its corresponding features.

A. Equilibrium Definition

We first define the support correspondence. Then, we define a firm's best response and the firm location equilibrium based on such a correspondence.

1) Support Correspondence: We define a correspondence that outputs a vector's positive elements.

Definition 2 (Support Correspondence)**.** *For a vector $\boldsymbol{z} \in \mathbb{R}_+^{1\times L}$, the support correspondence $S(\boldsymbol{z}) = \{s \in \mathcal{L} : z_s > 0\}$ is the set of indexes corresponding to positive elements in $\boldsymbol{z}$.*

For example, if $\boldsymbol{z} = \{3, 0, 2, 0\}$, then $S(\boldsymbol{z}) = \{1, 3\}$. Consequently, for any strategy $\boldsymbol{x}^p$, the correspondence $S(\boldsymbol{x}^p) = \{s \in \mathcal{L} : x_s^p > 0\}$ indicates the set of locations chosen by the firms preferring location p under the strategy $\boldsymbol{x}^p$.

2) Best Response: We now define a firm's best response.

For a firm preferring location $p \in \mathcal{L}$, its best response location choice is the set of locations that induce the maximum firm payoff, i.e.,

$$BR^p(\alpha, \boldsymbol{x}) = \arg\max_{s\in\mathcal{L}} F_s^p(\alpha, \boldsymbol{x}). \quad (7)$$

Normally, the best response is defined as a correspondence of all other firms' strategies excluding the firm its own's. However, due to the non-atomic firm assumption, the change of one firm's strategy does not affect the aggregate strategies of all the firms [17]. This allows us to directly write the best response as a correspondence of $\boldsymbol{x}$.

Based on a single firm's best response, we define a correspondence representing the aggregate best response of all the firms preferring location $p \in \mathcal{L}$:

$$ABR^p(\alpha, \boldsymbol{x}) = \{\boldsymbol{z} \in \mathbb{R}_+^{1\times L} : S(\boldsymbol{z}) \subset BR^p(\alpha, \boldsymbol{x}), \sum_{s\in\mathcal{L}} z_s \le N^p\}. \quad (8)$$

Specifically, the aggregate best response for the firms preferring a location p is any vector $\boldsymbol{z}$ such that all its elements (locations) with positive firm numbers belong to $BR^p(\alpha, \boldsymbol{x})$, i.e., $S(\boldsymbol{z}) \subset BR^p(\alpha, \boldsymbol{x})$, and its element sum is no larger that the total number of firms preferring location p, i.e., $\sum_{s\in\mathcal{L}} z_s \le N^p$. Notice that we allow firms with the same preference to choose different locations in their best responses.

3) Definition of Firm Location Equilibrium: Firm location equilibrium is defined as the fixed point of the best responses.

Definition 3 (Firm Location Equilibrium)**.** *Given any α, firm location strategy $\boldsymbol{x}$ is an equilibrium if and only if the aggregate strategy of each firm population preferring the same location p belongs to their aggregate best response under $\boldsymbol{x}$, i.e., $\boldsymbol{x}^p \in ABR^p(\alpha, \boldsymbol{x}), \forall p \in \mathcal{L}$.*

An interpretation of this equilibrium is that a strategy $\boldsymbol{x}$ is an equilibrium if and only if the firms' aggregate best response (i.e., the updated firm strategies under their best responses) can recover this strategy distribution $\boldsymbol{x}$. In other words, under the firm location equilibrium, no firm will change its location choice according to its best response.

B. Deriving the Firm Location Equilibrium

Directly computing the equilibrium based on the best response is challenging, due to the challenge of computing the fixed point of the multi-dimensional best response mapping of an $L \times L$-dimensional vector $\boldsymbol{x} = \{x_s^p, \forall p, s \in \mathcal{L}\}$. Instead of directly deriving the equilibrium distribution, we first prove that the Stage II game is a potential game. Under this, all the firms' payoffs can be related to the same function, i.e., the potential function, which allows us to characterize the equilibrium by solving an optimization problem. Then, we derive the firm location equilibrium.

The key proof of a potential game is to identify a potential function. However, there does not exist a general methodology for doing this, and we have to identify the potential function by exploiting the specific structure of the problem.

Lemma 1 (Stage II Game as Potential Game)**.** *Given any α, the Stage II game is a potential game with non-atomic players[6], which has a potential function*

$$f(\alpha, \boldsymbol{x}) = \alpha \times \sum_{s\in\mathcal{L}} \int_0^{\widehat{N}_s(\boldsymbol{x})} \frac{D(M^s, z)}{z} dz - V \times \sum_{s\in\mathcal{L}}\sum_{p\in\mathcal{L}} x_s^p - W \times \sum_{s\in\mathcal{L}}\sum_{p\in\mathcal{L}} x_s^p (l_p - l_s)^2. \quad (9)$$

Proof. According to [7], the Stage II game is a potential game if there is a potential function $f(\alpha, \boldsymbol{x})$ such that the following equality always holds:

$$\frac{\partial f(\alpha, \boldsymbol{x})}{\partial x_s^p} = F_s^p(\alpha, \boldsymbol{x}), \ \forall s, p \in \mathcal{L}. \quad (10)$$

By checking the first-order partial derivative of (9), we can show that the Stage II game is a potential game, and $f(\alpha, \boldsymbol{x})$ is the potential function. □

Showing that the game is a potential game allows us to characterize the Stage II firm location equilibrium by solving an optimization problem, which is easier than finding the fixed point of the firms' best responses. Formally,

[6] A potential game with non-atomic players is different from that with atomic players. For detailed discussions, see [7].

Theorem 1 (Firm Location Equilibrium)**.** *The set of firm location equilibria of the Stage II game is the set of global optimal solutions to the following optimization problem:*

$$\boldsymbol{x}^*(\alpha) \triangleq \arg\underset{\boldsymbol{x}\geq 0}{\text{maximize}} \quad f(\alpha, \boldsymbol{x}) \tag{11a}$$

$$\text{subject to} \quad \textstyle\sum_{s\in\mathcal{L}} x_s^p \leq N^p,\ \forall p \in \mathcal{L}. \tag{11b}$$

(STAGE II-NE)

More specifically, a vector $\boldsymbol{x}$ is an equilibrium, i.e., $\boldsymbol{x} \in \boldsymbol{x}^*(\alpha)$, if and only if there exists a pair of $\boldsymbol{\mu} \in \mathbb{R}^{1\times L}$ and $\boldsymbol{\lambda} \in \mathbb{R}^{L\times L}$ such that the following constraints are satisfied:

$$F_s^p(\alpha, \boldsymbol{x}) = \mu^p - \lambda_s^p, \qquad \forall p, s, \tag{12a}$$

$$\lambda_s^p x_s^p = 0,\ \lambda_s^p \geq 0,\ x_s^p \geq 0, \qquad \forall p, s, \tag{12b}$$

$$\left(\textstyle\sum_{s\in\mathcal{L}} x_s^p - N^p\right)\mu^p = 0,\ \mu^p \geq 0, \qquad \forall p, \tag{12c}$$

$$\textstyle\sum_{s\in\mathcal{L}} x_s^p \leq N^p, \qquad \forall p. \tag{12d}$$

(STAGE II-NE-CONDITION)

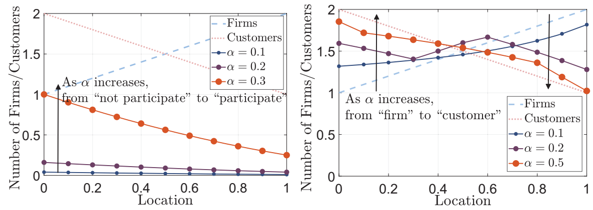
Proof. As proved in Lemma 1, this game is a potential game, so its equilibria are the solutions to Problem (STAGE II-NE) [7], i.e., maximizing the potential function $f(\alpha, \boldsymbol{x})$ under firm population constraints (i.e., among the firms who preferring any location $p \in \mathcal{L}$, the total number of participating firms should be no large than the total number of firms). On the other hand, the conditions (STAGE II-NE-CONDITION) are essentially the KKT conditions of Problem (STAGE II-NE). To show that (STAGE II-NE-CONDITION) are the conditions for equilibria, we have to show that the KKT conditions of Problem (STAGE II-NE) is necessary and sufficient conditions to its global optimal solutions. This is true because the $f(\alpha, \boldsymbol{x})$ in (11a) is concave (by checking its Hessian Matrix), and the constraint (11b) fulfills the Slater's condition. □

Note that the firm location equilibrium (derived from Theorem 1) may not be unique under a given α. Nevertheless, we can show that any of the equilibria leads to the same sets of firms' payoffs and the same platform's payoff,

Corollary 1 (Unique Equilibrium Payoffs)**.** *Under any given α, any equilibrium of the Stage II game induces the same set of firms' payoffs, i.e., $F_s^p(\alpha, \boldsymbol{x}) = F_s^p(\alpha), \forall p, s \in \mathcal{L}, \boldsymbol{x} \in \boldsymbol{x}^*(\alpha)$, and the same platform's payoff, i.e., $G(\alpha, \boldsymbol{x}) = G(\alpha), \forall \boldsymbol{x} \in \boldsymbol{x}^*(\alpha)$.*

Specifically, the dual variables of Problem (STAGE II-NE), i.e., $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$, are unique, because the constraints (11b) are linearly independent [18]. Hence, according to (12a), under a fixed α, any of the equilibria induces the same set of firms' payoffs, i.e., $F_s^p(\alpha, \boldsymbol{x}) = F_s^p(\alpha), \forall p, s \in \mathcal{L}, \boldsymbol{x} \in \boldsymbol{x}^*(\alpha)$. Based on this, we can show that under a fixed α, any of the equilibria induces the same set of aggregate number of firms at all locations, i.e., $\widehat{N}_s(\boldsymbol{x}) = \widetilde{N}_s(\alpha), \forall s \in \mathcal{L}, \boldsymbol{x} \in \boldsymbol{x}^*(\alpha)$. This is because the mapping from the firms' payoffs (defined in (4)) to aggregate number of firms is a one-to-one correspondence, due to the strictly increasing donation function in the number of firms as in Assumption 1. Hence, from the platform's point of view, given any α, it achieves the same payoff under any of the firm location equilibria in Stage II, as its payoff (defined in (3)) only depends on the aggregate number of firms $\widehat{N}_s(\boldsymbol{x})$.

In the rest of this paper, let $G(\alpha)$ denote the platform's payoff under the firm location equilibria given an α.



(a) (b)

Fig. 4. The impact of α on the firm location equilibrium.

C. Impact of α on Firm Location Equilibrium

Based on the conditions in Theorem 1, we show how the firm location equilibrium changes with the DSF α. A key insight is that a larger α leads to more firm participations and a better match to the customers' preferences.

Given any opportunity cost V, deviation cost W, firm preference $N^s, \forall s \in \mathcal{L}$, and customer preference $M^s, \forall s \in \mathcal{L}$, the firm location equilibrium changes with α as follows.

Proposition 1 (Participation and Preference Matching)**.** *The ratios W/α and V/α determine the Stage II equilibria, i.e.,*

- *$V/\alpha \to 0$: full participation, $\sum_{s\in\mathcal{L}} x_s^p = N^p, \forall p \in \mathcal{L}$;*
- *$V/\alpha \to \infty$: no participation, $\sum_{s\in\mathcal{L}} x_s^p = 0, \forall p \in \mathcal{L}$;*
- *$W/\alpha \to 0$: full preference matching[7],*

$$U_s(\widehat{N}_s(\boldsymbol{x})) = \frac{D(M^s, \widehat{N}_s(\boldsymbol{x}))}{\widehat{N}_s(\boldsymbol{x})} = \widetilde{U}, \forall s \in \mathcal{L}, \tag{13}$$

  where $\widetilde{U}$ is a positive value;
- *$W/\alpha \to \infty$: no active matching, $\sum_{s\in\mathcal{L}/p} x_s^p = 0, \forall p \in \mathcal{L}$.*

Specifically, the potential function (9) is a weighted sum of three functions with the corresponding weights as α, V, and W, respectively. As these weights change, the firm location equilibrium (the optimal solution to Problem (STAGE II-NE)) changes accordingly as in Proposition 1.

Figure 4 shows the impact of α under two choices of (W, V). Here we choose the donation function $D(M, N) = M\sqrt{N}$. The x-axis represents the location, and the y-axis corresponds to the number of firms or customers. The "Firm" and "Customer" curves correspond to the firms' and customers' location preferences, respectively. The curves labeled with $\alpha = 0.1$, 0.2, and 0.3 are the firm location equilibrium under the corresponding values of α.

Figure 4 (a) shows the results with $W = 0$ and $V = 1$, under which firms always fully match the customers' preferences due to the zero deviation cost W. In this case, under any α, firms

[7]We refer this case as "full preference matching", because the firms gain the same average donations, i.e., $\widetilde{U}$, at all locations (under the deviation), so that further deviation cannot increase their payoffs.

are distributed in a shape that is similar as the customers' preferences do. As α increases, the firm participation increases, i.e., the total number of participating firms increases. Figure 4 (b) shows the result with $W = 1$ and $V = 0$, under which firms always fully participate due to the zero opportunity cost V. In this case, as α increases, the firm matching increases, i.e., firms' location choices deviate from firms' preferences (i.e., the blue dash line) to match customers' preferences (i.e., the red dot line). To sum up, a larger α leads to more firm participation and a better match to the customers' preferences.

V. Stage I: Platform DSF Decision

In Stage I, the platform chooses the DSF α to maximize its payoff under the firm location equilibrium. We first present the platform's payoff optimization problem. As the problem is non-convex and cannot be solved in closed-form, we derive the upper-bound and lower-bound of the optimal solution, and propose an exhaustive searching method.

A. Platform Profit Maximization Problem

In Stage I, the platform selects the optimal fraction α^* that maximizes its payoff. Formally,

$$\alpha^* \triangleq \arg \max_{\alpha \in [0,1]} \quad G(\alpha) \qquad \text{(STAGE I-NE)}$$

Here $G(\alpha)$ is the platform's payoff under the firm location equilibrium given an α, as in Corollary 1 in Section IV-B.

Problem (STAGE I-NE) is a non-convex optimization problem due to the non-convex objective function $G(\alpha)$. Specifically, the objective function $G(\alpha)$ is a piece-wise function that is not always differentiable. In addition, this piece-wise function may not be a quasi-concave function, so we cannot use an effective bisection algorithm [19] to solve the problem. Hence, it is difficult to derive the closed-form optimal solution to Problem (STAGE I-NE).

B. Optimal Solution Bounds and Approximate Solution

Despite the non-convexity, we can characterize the upper- and lower-bound of the optimal solution to Problem (STAGE I-NE). Within the bounds, we can then implement an exhaustive searching to obtain an approximate optimal solution.

1) Upper-Bound and Lower-Bound of α^*: The upper-bound and lower-bound of α^* are as follows:

Proposition 2 (Upper-Bound and Lower-Bound of α^*)**.** *The optimal α^* is upper-bounded by $\overline{\alpha}$ that satisfies*

$$\overline{\alpha} = \max\{\eta_{M^p}(\overline{x}^p), \forall p \in \mathcal{L}\}, \tag{14}$$

where $\overline{x}^p = \{x | [D(M^p, x)]_x = V\}$ for all $p \in \mathcal{L}$, and $\eta_M(N)$ is defined in (1)*.*

The optimal α^* is lower-bounded by $\underline{\alpha}$ as follows

$$\underline{\alpha} = \min\left\{\frac{V}{\max\{\frac{D(M^p,N^p)}{N^p}, \forall p \in \mathcal{L}\}}, \hat{\alpha}\right\}. \tag{15}$$

where $\hat{\alpha}$ is defined as

$$\sum_{s \in \mathcal{L}} \frac{D(M^s, x_s(\hat{\alpha}))}{1 - \eta_{M^s}(x_s(\hat{\alpha}))} \left(\frac{\eta_{M^s}(x_s(\hat{\alpha}))}{\hat{\alpha}} - 1\right) = 0. \tag{16}$$

Specifically, the upper-bound $\overline{\alpha}$ is characterized by the elasticity of the customers' donations: when the elasticity is larger, the customers' donations are more sensitive to the number of firms, so the platform should increase α^* to incentivize firms to participate and satisfy customers' preferences. The lower-bound $\underline{\alpha}$ is characterized by how hard it is to motivate participation: if either the opportunity cost V is larger or the $\max\{D(M^p, N^p)/N^p, \forall p \in \mathcal{L}\}$ is smaller,[8] the lower bound is larger (which means a larger incentive is needed to motivate firm participation). When the participation motivation is quite hard (i.e., when $\hat{\alpha}$ is effective in (15)), the platform maintains a certain level of participation motivation (i.e., $\hat{\alpha}$).

2) Searching Method: The last step of computing the optimal DSF is to search in the interval of $[\underline{\alpha}, \overline{\alpha}]$. Specifically, we divide the internal into K segments, and the approximate optimal solution is $\alpha_K^* = \arg\max\{G(\alpha) | \alpha \in \{\underline{\alpha} + (\overline{\alpha} - \underline{\alpha})k/(K-1), k = 0, 1, ..., K-1\}\}$. Let $\alpha^* = \arg\max_{\alpha \in [0,1]} G(\alpha)$ be the actual optimal solution. The gap $|G(\alpha_K^*) - G(\alpha^*)|$ is bounded:

Lemma 2 (Optimal Solution Approximation)**.** *Given any ϵ, there always exists a threshold $\underline{K}$ such that $|G(\alpha_K^*) - G(\alpha^*)| \le \epsilon$ for any $K \ge \underline{K}$.*

The proof of Lemma 2 relies on applying the Maximum Theorem [20] to show the continuity of function $G(\alpha)$.

VI. Case Study with Data Collected from Twitch

We collect real-world data from Twitch platform and conduct analysis based on our model accordingly. The data is collected from Twitch every 15 minutes from Nov. 05 to Nov. 20, 2017. The information contains user_id, game_id, streamer_type, viewer_count, started_at, and language.

In this case study, we demonstrate how to compute the platform's optimal DSF with only the firms' and customers' actual behaviors data (instead of their preferences, which are usually private information). Based on the case study, we also suggest that under the collected data and our model settings, Twitch should significantly reduce the value of DSF (comparing with its current practice) to enhance its payoff.

We focus on the game *League of Legends*, and consider the streaming time as the service attribute. To better capture the periodic feature of the time attribute and the streamers' multiple time slots streams, we propose a new attribute model, i.e., a circular model with multiple location coverage. Although this model is different (and more complicated) from the unit length interval model discussed in Section III, our modeling and analysis are still applicable (detailed discussions in VI-A).

We first discuss the system setting, then explain how to map from the firms' choices of streaming time (observable from the collected data) to their time preferences (not directly observable). Finally, we derive the optimal DSF.

[8] Intuitively, a smaller $\max\{D(M^p, N^p)/N^p, \forall p \in \mathcal{L}\}$ implies a smaller average donation after full participation, under which it is harder to motivate the full participation.

A. *System Setting*

1) New Attribute Model: The model here is different from the previous model (in Section III) in two aspects. (i) Circular model: the attributes are distributed over a circle (with no extreme values) instead of over a line interval (with two extreme values). (ii) Multiple location coverage: once a firm selects a location, its service can cover several locations (starting from the selected one). Let $\mathcal{L}_s \subset \mathcal{L}$ denote the set of locations that a firm can cover if it selects a location $s \in \mathcal{L}$. Figure 5 illustrates such a model, where the circle represents 24 hours in a day. More specifically, the circle contains 96 locations, so the distance between each pair of adjacent locations corresponds to 15 minutes (which corresponds to the time interval between two consecutive data collections in our dataset). We assume that each streamer broadcasts for an consecutive period of 2 hours, which corresponds to the average broadcasting time of streamers in *League of Legends*. For example, when a streamer selects 1am (location 4), he will continue to serve until 3am (location 12), represented by the shaded area in Figure 5, i.e., $\mathcal{L}_4 = \{4, 5, ..., 12\}$.

Under this new attribute model, if a firm preferring location $p \in \mathcal{L}$ participates and chooses a location $s \in \mathcal{L}$, its payoff is the difference between the donation gain over all the locations in set $\mathcal{L}_s$ and its cost in the circular model, i.e.,

$$\widetilde{F}_s^p(\alpha, \boldsymbol{x}) = \alpha \times \textstyle\sum_{l \in \mathcal{L}_s} U(M^l, \widehat{N}_l(\boldsymbol{x})) - \widetilde{C}_s^p(V, W), \quad (17)$$

where the cost in the circular model $\widetilde{C}_s^p(V, W)$ is the sum of the opportunity cost and the deviation cost that is associated the shortest path along the circle, i.e.,

$$\widetilde{C}_s^p(V, W) = V + W \times (\min\{|l_p - l_s|, 1 - |l_p - l_s|\})^2. \quad (18)$$

The firms' non-participation payoffs and the platform's payoff are the same as those of the line model in Section III.

Under this new model, we can still prove that the Stage II game is a potential game, just with a different and more complicated potential function $\widetilde{f}(\alpha, \boldsymbol{x})$.

Lemma 3 (Firm Location Equilibrium Under the New Attribute Model)**.** *Given any α, the Stage II game is a potential game with non-atomic players, with a potential function*

$$\widetilde{f}(\alpha, \boldsymbol{x}) = \alpha \times \left(\sum_{s \in \mathcal{L}} \sum_{l \in \mathcal{L}_s} \int_0^{\sum_{h \in L(l)} \sum_{p \in \mathcal{L}} x_h^p} \frac{D(M^l, z)}{z \times |L(l)|} dz \right) - \sum_{s \in \mathcal{L}} \sum_{p \in \mathcal{L}} x_s^p \widetilde{C}_s^p(V, W), \quad (19)$$

where $L(l) \triangleq \{h | l \in \mathcal{L}_h\}$ is the location strategy set that can cover location l, and $|L(l)|$ is the size of the set $L(l)$.

The set of firm location equilibria of the Stage II game is the set of global optimal solutions of the following problem:

$$\boldsymbol{x}_{cir}^*(\alpha) \triangleq \arg\underset{\boldsymbol{x} \geq 0}{\text{maximize}} \quad \widetilde{f}(\alpha, \boldsymbol{x}), \quad (20a)$$

$$\text{subject to} \quad \textstyle\sum_{s \in \mathcal{L}} x_s^p \leq N^p, \ \forall p \in \mathcal{L}. \quad (20b)$$

Accordingly, we can obtain a similar result as it in Corollary 1. That is, under given any fixed α, all Stage II equilibria lead to the same sets of firms' payoffs and the same platform's payoff. Hence, in Stage I, we can compute the optimal DSF by searching over an interval of α as in Section V-B2.

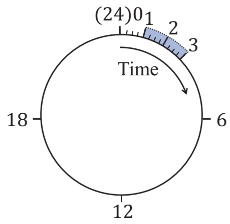


Fig. 5. Firm model.

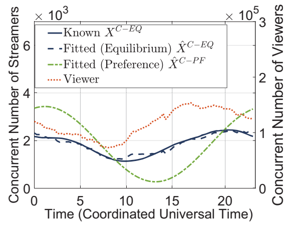


Fig. 6. A fitting example.

2) Donation Function: As Twitch does not provide donation information through API, we choose the donation function according to paper [9], which analyzes the donation from Douyu[9]. In [9], the donation to a live stream increases with the number of viewers in the following manner:

$$[\text{received donation per firm}] = e^{b_0}([\text{viewers per firm}])^{b_1}, \quad (21)$$

where $b_0 = -1.17$ and $b_1 = 0.6$ based on empirical data. Hence, we use the following donation function:

$$D(M, N) = e^{b_0} (M/N)^{b_1} N, \quad (22)$$

which is the per-firm donation $e^{b_0}(M/N)^{b_1}$ multiplied by the number of firms. This $D(M, N)$ satisfies Assumption 1.

3) Current Donation-Split-Fraction: On Twitch, viewers purchase 100 bits (i.e., a virtual currency on Twitch) with \$1.4, while streamers can exchange 100 bits with \$1. Hence, $\alpha = 1/1.4 \approx 0.71$.

B. *Mapping from Firm Distribution to Firm Preference*

Before deriving the firms' and platform's equilibrium strategies, we need to first estimate the firms' preference locations based on the firms' and customers' actual locations.

Specially, based on the known cumulative firm distribution $\boldsymbol{X}^{\text{C-EQ}}$ (i.e., how many firms serving customers at each location in the dataset, assuming that these firms behave according to the equilibrium in Lemma 3), we aim to estimate the unknown actual firm preference distribution $\boldsymbol{X}^{\text{PF}}$ (i.e., how many firms prefer to start at each location). The consideration of "cumulative" is due to fact that a streamer will cover multiple locations. The estimated firm preference is denoted by $\hat{\boldsymbol{X}}^{\text{PF}}$, based on which we can obtain the cumulative firm (preference) distribution $\hat{\boldsymbol{X}}^{\text{C-PF}}$ (i.e., how many firms serving customers at each location if all the firms start at their preferring locations), and the cumulative firm (equilibrium) distribution $\hat{\boldsymbol{X}}^{\text{C-EQ}}$ (which will be different from $\boldsymbol{X}^{\text{C-EQ}}$ due to the errors introduced in the estimation process). Each of the vectors defined above has 96 elements, where the s^{th} element corresponds to the number of firms at location s.

We assume that the firm actual preference $\boldsymbol{X}^{\text{PF}}$ follows a sine function, i.e., the number of firms at a location $l \in \mathcal{L}$ is

[9]Douyu is one of the most popular live streaming platforms in China, and it has a similar business model as Twitch.

TABLE II
OPTIMAL DSF.

$V\backslash W$	2	10	20	100	200	1000
0.2	0.08	0.09	0.08	0.07	0.07	0.07
1	0.32	0.31	0.30	0.28	0.28	0.28
2	0.39	**0.38**	0.38	0.40	0.40	0.40
4	0.40	0.40	0.40	0.40	0.40	0.40

$S(l) = c_1 \times sin(2\pi l + c_2) + c_3$. This is because the lag plot of $\boldsymbol{X}^{\text{C-EQ}}$ follows a circular shape, which suggests that the $\boldsymbol{X}^{\text{C-EQ}}$ is in a sine function [21]. Moreover, we can verify through simulation that a sine function preference $\boldsymbol{X}^{\text{PF}}$ is likely to output a sine function equilibrium $\boldsymbol{X}^{\text{C-EQ}}$.

To estimate $\boldsymbol{X}^{\text{PF}}$, the key is to estimate the set of parameters (c_1, c_2, c_3). We choose the parameters that minimize the root-mean-square-error (RMSE) between the known cumulative distribution $\boldsymbol{X}^{\text{C-EQ}}$ and fitted cumulative distribution $\hat{\boldsymbol{X}}^{\text{C-EQ}}$:

$$\text{RMSE} = \sqrt{\sum_{l \in \mathcal{L}} \left(X_l^{\text{C-EQ}} - \hat{X}_l^{\text{C-EQ}} \right)^2 / L}. \tag{23}$$

Figure 6 shows the fitting result (under the parameters (c_1, c_2, c_3) that lead to the minimum RMSE).[10] The "Fitted (Equilibrium) $\hat{X}^{\text{C-EQ}}$" is the fitted cumulative firm distribution in equilibrium, and the "Fitted (Preference) $\hat{X}^{\text{C-PF}}$" is the fitted cumulative firm distribution when all the firms choose their preferring locations. We can see that the streamers deviate from "Fitted (Preference) $\hat{\boldsymbol{X}}^{\text{C-PF}}$" to "Fitted (Equilibrium) $\hat{X}^{\text{C-EQ}}$" to better match the viewers' preferences distribution.

C. Deriving Optimal Donation-Split-Fraction

Based on the estimated $\hat{\boldsymbol{X}}^{\text{PF}}$, we derive the platform's optimal DSF based on Lemma 3 and the numerical search mentioned in Section V-B2. Table II shows the optimal DSF values under different possible values of V and W (as we do not know the actual values). In this table, we use bold fonts to represent the values corresponding to $V = 2$ and $W = 10$, because this combination of parameters leads to the minimum RMSE (defined in (23)) over all the possible values (hence is most likely to be the one in reality).

Remark 1 (Optimal α^*)**.** *In Table II, the bold text suggests that the optimal DSF should be $\alpha^* = 0.38$. Furthermore, all the optimal DSF values under various V and W values are significantly smaller than 0.71 chosen by Twitch.*

VII. CONCLUSION

This paper studies the platform's optimal donation-split-fraction (DSF) choice and the firms' equilibrium service attribute selections in a donation-based market. Our analysis shows that, regarding the firm service attribute selection, a larger DSF leads to more firm participations and a better match to the customers' preferences. Regarding the platform's optimal DSF, we derive the upper-bound and lower-bound of the optimal DSF. In addition, we perform a case study based on the dataset from Twitch. Our analysis and simulation results suggest that there exists a significant potential for Twitch to improve its revenue, by setting the DSF to 0.38, instead of 0.71 as in Twitch's current practice.

This work serves as a first step towards understanding the operation and revenue management in donation-based market. There are several potential interesting extensions. First, we can further consider the strategic behaviors of the customers in terms of their donations and attribute deviations. Second, it is interesting to consider the non-equal donation sharing among firms, to understand the different location selection behaviors of the firms with different qualities.

[10] The probability density function of the fitting residuals roughly follows a normal distribution with zero mean. This implies that the residuals are random, which suggests that our fitting model works well [21].

REFERENCES

[1] K. Scheibe, K.J. Fietkiewicz, and W.G. Stock, "Information behavior on social live streaming services," *JISTaP*, vol. 4, no. 2, pp. 6-20, 2016.

[2] A. Hicks, "Streamlabs livestreaming Q4 report," streamlabs.com, Jan. 25, 2018. [Online]. Available: https://blog.streamlabs.com/streamlabs-livestreaming-q4-report-tipping-reaches-100m-for-the-year-youtube-dominates-in-4bf450fae536. [Accessed Dec. 7, 2018].

[3] Twitch, "2017 year in review," twitch.tv, 2018. [Online]. Available: https://www.twitch.tv/year/2017/. [Accessed Dec. 7, 2018].

[4] C. d'Aspremont, J.J. Gabszewicz, and J.F. Thisse, "On Hotelling's" stability in competition," *Econometrica*, vol. 47, no. 5, pp. 1145-1150, 1979.

[5] H.A. Eiselt and V. Marianov, "Foundations of location analysis", *Springer Science & Business Media*, 2011.

[6] M.A. Khan and Y. Sun, "Non-cooperative games with many players," *Handbook of Game Theory with Economic Applications*, vol. 3, pp. 1761-1808, 2002.

[7] W.H. Sandholm, "Potential games with continuous player sets," *J. Econ. Theory*, vol. 97, no. 1, pp. 81-108, 2001.

[8] M. Hu, M. Zhang, and Y. Wang, "Why do audiences choose to keep watching on live video streaming platforms? An explanation of dual identification framework," *Comput. Human Behav.*, vol. 75, pp. 594-606, 2017.

[9] Z. Zhu, Z. Yang, and Y. Dai, "Understanding the gift-sending interaction on live-streaming video websites," *In Proc. SCSM*, pp. 274-285, 2017.

[10] P. Tang, Y. Zeng, and S. Zuo, "Fans economy and all-pay auctions with proportional allocations," *In Proc. AAAI*, pp. 713-719, 2017.

[11] A.L. Jia, S. Shen, D.H.J. Epema, and A. Iosup, "When game becomes life: The creators and spectators of online game replays and live streaming," *ACM TOMM*, vol. 12, no. 4, pp. 47, 2016.

[12] H. Hotelling, "Stability in competition," *The Collected Economics Articles of Harold Hotelling*, NY: Springer, pp. 50-63, 1990.

[13] N. Economides, "Minimal and maximal product differentiation in Hotelling's duopoly," *Economics Letters*, vol. 21, no. 1, pp. 67-71, 1986.

[14] S.Brenner, "Hotelling games with three, four, and more players," *J. Reg. Sci.*, vol. 45, no. 4, pp. 851-864, 2005.

[15] S. Behringer and L. Filistrucchi, "Hotelling competition and political differentiation with more than two newspapers," *Information Econ. & Policy*, vol. 30, pp. 30: 36-49, 2015.

[16] D. Schmeidler, "Equilibrium points of nonatomic games," *J. Stat. Phys.*, vol. 7, no. 4, pp. 295-300, 1973.

[17] R.J. Aumann and L.S. Shapley, "Values of non-atomic game," *Princeton University Press*, 2015.

[18] J. Kyparisis, "On uniqueness of Kuhn-Tucker multipliers in nonlinear programming," *Math. Program.*, vol. 32, no. 2, pp. 242-246, 1985.

[19] R.L. Burden and J.D. Faires, "Numerical analysis," *Brooks/Cole*, 2001.

[20] E. Cech, Z. Frolik, M. Katetov, "Topological spaces," 1966.

[21] NIST/SEMATECH e-Handbook of Statistical Methods, http://www.itl.nist.gov/div898/handbook/, 2018.